Quantum many-body dynamics in a Lagrangian frame. I. Equations of motion and conservation laws.

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We formulate equations of motion and conservation laws for a quantum many-body system in a co-moving Lagrangian reference frame. It is shown that generalized inertia forces in the co-moving frame are described by Green's deformation tensor $g_{\mu\nu}(\boldsymbol{\xi}, t)$ and a skew-symmetric vorticity tensor $\widetilde{F}_{\mu\nu}(\boldsymbol{\xi}, t)$, where $\boldsymbol{\xi}$ in the Lagrangian coordinate. Equations of motion are equivalent to those for a quantum many-body system in a space with time-dependent metric $g_{\mu\nu}(\boldsymbol{\xi}, t)$ in the presence of an effective magnetic field $\widetilde{F}_{\mu\nu}(\boldsymbol{\xi}, t)$. To illustrate the general formalism we apply it to the proof of the harmonic potential theorem. As another example of application we consider a fast long wavelength dynamics of a Fermi system in the dynamic Hartree approximation. In this case the kinetic equation in the Lagrangian frame can be solved explicitly. This allows us to formulate the description of a Fermi gas in terms of an effective nonlinear elasticity theory. We also discuss a relation of our results to time-dependent density functional theory.

I. INTRODUCTION

Lagrangian and Eulerian formulations of fluid mechanics are known as two alternative ways to describe dynamics of continuum media[1]. The more common Eulerian (or spatial) formulation considers basic collective variables, such as density $n(\mathbf{x}, t)$ or current $\mathbf{j}(\mathbf{x}, t)$ distributions, as functions of space-time coordinates $\mathbf{x}$ and t[1,2]. This corresponds to the description of a system from the standard point of view of an observer in a fixed laboratory reference frame. Central notions of Lagrangian (or material) description are the trajectories of infinitesimal fluid elements. Every small element of a fluid can be uniquely labeled by its initial position $\boldsymbol{\xi}$ that plays a role of independent, so called Lagrangian, coordinate. Lagrangian description represents the dynamics of continuum media as it is seen by a local observer, moving with a flow. In the last decades the Lagrangian method attracts an increasing attention as a powerful tool for studying nonlinear dynamics of compressible media with numerous applications in cosmology, plasma physics, physics of semiconductors, etc. (for a recent comprehensive review see Ref. 3). Recently we have shown that the Lagrangian coordinate naturally appears in time-dependent density functional theory (TDDFT), where it plays a role of a basic variable for a nonadiabatic exchange correlation potential[4]. It is also interesting to note a relation of Lagrangian fluid dynamics to noncommutative geometry and noncommuting gauge fields[5].

Commonly Lagrangian and Eulerian descriptions are considered as inherent ingredients of the classical continuum mechanics. In fact, they offer two alternative techniques for solving the equations of classical hydrodynamics. However, the main idea of Lagrangian method, which is the description of dynamics using co-moving coordinates, is clearly much more general and universal. In the present paper we formulate microscopic equations of many-body dynamics in the co-moving Lagrangian reference frame. The transformation to the Lagrangian frame corresponds to an explicit separation of the convective motion of particles. This is a natural generalization of the common separation of the center-of mass motion in homogeneous many-body systems. The separation of the center-of-mass motion also plays an important role in the theory of harmonically trapped systems. For the harmonic inhomogeneity the convective motion can be separated by the transformation to a global accelerated reference frame, which is a key step in the proof of the harmonic potential theorem[6,7,8] (HPT). In fact, the proof of HPT can be viewed as the simplest application of the Lagrangian description to quantum dynamics. In the case of a general inhomogeneous flow the separation of convective "center-of-mass" motion leads to an appearance of inhomogeneous inertia forces in the equations for the relative motion. We show that these forces can be uniquely described by the symmetric deformation tensor $g_{\mu\nu}(\boldsymbol{\xi}, t)$ and a skew-symmetric vorticity tensor $\widetilde{F}_{\mu\nu}(\boldsymbol{\xi}, t)$. The deformation tensor enters equations of many-body dynamics as an effective time-dependent metric, while the vorticity tensor plays a role of an effective magnetic field.

A great advantage of the Lagrangian description of many-body dynamics is that in the co-moving frame both the density of particles and the current density become the exact integrals of motion. The current density is zero in every point of Lagrangian $\boldsymbol{\xi}$-space, while the particles' density distribution preserves its initial form. These "conservation laws" are guaranteed by a fine local compensation of inertia forces, external forces, and the force of internal stresses. The above force balance follows the local momentum conservation law (the exact microscopic Navier-Stokes equation) after the transformation to the

Lagrangian frame. We explicitly demonstrate that the exact internal stress force takes a form of a covariant divergence of a symmetric second-rank stress tensor. As a byproduct of our formalism we obtain a microscopic representation for the local stress tensor in a general quantum many-body system.

The concept of quantum stress has been introduced by Schrödinger in 1927[9]. Over the last two decades there has been a growing interest in understanding properties of quantum systems, such as molecules or solids, in terms of the stress density (see, for example, Refs. 10,11,12,13 and references therein). A derivation of a microscopic expression for the kinetic part of the stress tensor in quantum many-body system causes no problem. This simple generalization of the one-particle result has been obtained in the classical paper by Martin and Schwinger[14]. However, the derivation of the microscopic form for the interaction related stress tensor turned out to be not that simple[11,14,15,16,17,18]. In this paper we present two alternative derivations of the symmetric form for the stress density, which has been obtained by Puff and Gillis in Ref. 17. In particular, we show that this form is consistent with the definition of the stress tensor via the variational derivative of the energy with respect to the metric tensor.

The structure of the paper is following. In Sec. II we consider the standard Eulerian form of the conservation laws in a quantum many-body system. In this section we also present a compact derivation of the microscopic expression for the exact stress tensor. Section III is devoted to the formulation of quantum many-body theory in the co-moving Lagrangian frame. In Sec. IIIA the key notions of Lagrangian coordinate and of the deformation tensor are formally defined. The derivation of the equations of motion in an arbitrary local noninertial reference frame is presented in Sec. IIIB. Here we also derive the form of transformed many-body Hamiltonian and discuss the physical meaning of generalized inertia forces. In Sec. IIIC we derive local conservation laws, and present a complete formulation of the many-body problem in the Lagrangian frame. It is shown that this problem corresponds to the solution of the equations of motion for the relative motion, supplemented by the local force balance equation. The force balance equation plays a role of an additional gauge condition that fixes the reference frame. Section IV contains simple examples of application of the general theory. In Sec. IVA we interpret the harmonic potential theorem[6] in terms of dynamics in the Lagrangian frame. In Sec. IVB we apply the general formalism to the study of semiclassical collisionless dynamics of a Fermi gas, and shortly discuss a connection of our approach to TDDFT. It is shown that in the regime of a fast long wavelength evolution the kinetic equation in the Lagrangian frame can be solved explicitly. In this case the behavior of the system is described by an effective nonlinear continuum mechanics, which, after the transformation to the laboratory frame, reduces to the generalized collisionless hydrodynamics of

Refs. 19,20. In Sec. V we summarize our results.

II. CONSERVATION LAWS IN THE LABORATORY REFERENCE FRAME: DEFINITION OF THE STRESS TENSOR

In this paper we consider a system of N interacting particles in the presence of a time-dependent external potential $U_{\text{ext}}(\mathbf{x}, t)$. The corresponding Hamiltonian takes the following standard form

$$H = \widehat{T} + \widehat{W} + \widehat{U}, \tag{1}$$

$$\widehat{T} = -\int d\mathbf{x}\psi^\dagger(\mathbf{x})\frac{\nabla^2}{2m}\psi(\mathbf{x}), \tag{2}$$

$$\widehat{W} = \frac{1}{2}\int d\mathbf{x}d\mathbf{x}'w(|\mathbf{x}-\mathbf{x}'|)\psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{x}')\psi(\mathbf{x}')\psi(\mathbf{x}), \tag{3}$$

$$\widehat{U} = \int d\mathbf{x}U_{\text{ext}}(\mathbf{x}, t)\psi^\dagger(\mathbf{x})\psi(\mathbf{x}), \tag{4}$$

where $w(x)$ is the interaction potential, and $\psi^\dagger$ and ψ are the field operators, which satisfy proper commutation relations

$$[\psi^\dagger(\mathbf{x}), \psi(\mathbf{x}')]_\pm = \delta(\mathbf{x}-\mathbf{x}'). \tag{5}$$

The upper (lower) sign in Eq. (5) corresponds to fermions (bosons), and $[A, B]_\pm = AB \pm BA$. Using Hamiltonian of Eqs. (1)–(4) we obtain Heisenberg equations of motion for ψ-operators

$$i\frac{\partial}{\partial t}\psi(\mathbf{x}) = -\frac{\nabla^2}{2m}\psi(\mathbf{x}) + U_{\text{ext}}\psi(\mathbf{x})$$
$$+ \int d\mathbf{x}'w(|\mathbf{x}-\mathbf{x}'|)\psi^\dagger(\mathbf{x}')\psi(\mathbf{x}')\psi(\mathbf{x}) \tag{6}$$

Equation (6) allows to derive equations of motion for any physical observable as well as for any correlation function. The most important of these equations are the local conservation laws or balance equations, which should be satisfied for an arbitrary evolution of the system. Below we concentrate on conservation laws for the number of particles and for momentum. These local conservation laws follow the equations of motion for the density, $n(\mathbf{x}, t)$, and for the current, $\mathbf{j}(\mathbf{x}, t)$ respectively. Computing the time derivative of the density operator,

$$\widehat{n}(\mathbf{x}, t) = \psi^\dagger(\mathbf{x}, t)\psi(\mathbf{x}, t), \tag{7}$$

we obtain the continuity equation that is the local balance equation for the number of particles

$$\frac{\partial n}{\partial t} + \frac{\partial j_\mu}{\partial x^\mu} = 0, \tag{8}$$

where $n(\mathbf{x}, t) = \langle\widehat{n}(\mathbf{x}, t)\rangle$ and

$$j_\mu(\mathbf{x}, t) = \langle\widehat{j}_\mu(\mathbf{x}, t)\rangle = -\frac{i}{2m}\left\langle\psi^\dagger\frac{\partial\psi}{\partial x^\mu} - \frac{\partial\psi^\dagger}{\partial x^\mu}\psi\right\rangle. \tag{9}$$

Here the angle brackets stand for averaging with the exact many-body density matrix. Similarly using Eq. (6) we derive the equation of motion for the current, Eq. (9) (see, for example, Refs. 14,18)

$$m\frac{\partial j_\mu}{\partial t} + F_\mu^{\text{kin}} + F_\mu^{\text{int}} + n\frac{\partial}{\partial x^\mu}U_{\text{ext}} = 0, \qquad (10)$$

Equation (10) has a meaning of the local force balance equation in the fixed laboratory reference frame. Vectors F_μ^{kin} and F_μ^{int} in Eq. (10) correspond to the forces, which are related to the kinetic and the interaction effects respectively

$$F_\mu^{\text{kin}} = \frac{\partial}{\partial x^\nu}\frac{1}{2m}\left\langle \frac{\partial \psi^\dagger}{\partial x^\mu}\frac{\partial \psi}{\partial x^\nu} + \frac{\partial \psi^\dagger}{\partial x^\nu}\frac{\partial \psi}{\partial x^\mu} - \frac{\delta_{\mu\nu}}{2}\nabla^2\widehat{n}\right\rangle (11)$$

$$F_\mu^{\text{int}} = \int d\mathbf{x}'\frac{\partial w(|\mathbf{x}-\mathbf{x}'|)}{\partial x^\mu}\rho_2(\mathbf{x},\mathbf{x}') \qquad (12)$$

In Eq. (12) we introduced the notation $\rho_2(\mathbf{x},\mathbf{x}') = \langle\psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{x}')\psi(\mathbf{x}')\psi(\mathbf{x})\rangle$ for the two-particle density matrix. Obviously, the last term in the left hand side in Eq. (10) is the force produced by the external potential. The kinetic force of Eq. (11) has a form of a divergence of a symmetric second rank tensor. This automatically implies vanishing integral kinetic force, $\int F_\mu^{\text{kin}}(\mathbf{x},t)d\mathbf{x} = 0$. The Newton's third law requires that the force F_μ^{int} of Eq. (12) should obey the same property, which is however by far not obvious. In fact, the possibility to represent Eq. (12) in a divergence form has been a subject of a long discussion in the literature[11,14,16,17,18]. An elegant symmetric representation of the stress tensor has been presented (unfortunately without derivation) by Puff and Gillis in Ref. 17. Since this representation is of primary importance for our paper, below we give a compact derivation of the Puff and Gillis result.

The symmetry of the function $\rho_2(\mathbf{x},\mathbf{x}')$ with respect to the permutation of coordinates allows us to transform vector F_μ^{int}, Eq. (12), as follows

$$F_\mu^{\text{int}}(\mathbf{x}) = \int d\mathbf{x}'\frac{\partial w(|\mathbf{x}-\mathbf{x}'|)}{\partial x^\mu}\rho_2(\mathbf{x},\mathbf{x}')$$

$$= \frac{1}{2}\int d\mathbf{x}'\frac{\partial w(|\mathbf{x}'|)}{\partial x'^\mu}[\rho_2(\mathbf{x}-\mathbf{x}',\mathbf{x}) + \rho_2(\mathbf{x},\mathbf{x}-\mathbf{x}')]$$

$$= -\frac{1}{2}\int d\mathbf{x}'\frac{\partial w(|\mathbf{x}'|)}{\partial x'^\mu}[\rho_2(\mathbf{x}+\mathbf{x}',\mathbf{x}) - \rho_2(\mathbf{x},\mathbf{x}-\mathbf{x}')]$$

$$= -\frac{1}{2}\int d\mathbf{x}'\left[e^{x'^\nu\partial_\nu} - 1\right]\frac{\partial w(|\mathbf{x}'|)}{\partial x'^\mu}\rho_2(\mathbf{x},\mathbf{x}-\mathbf{x}')$$

where $\partial_\nu = \frac{\partial}{\partial x_\nu}$. Using an obvious operator identity

$$e^{\mathbf{x}'\nabla} - 1 = \int_0^1 \mathbf{x}'\nabla e^{\lambda\mathbf{x}'\nabla}d\lambda$$

we arrive at the following final representation for the local force F_μ^{int}

$$F_\mu^{\text{int}}(\mathbf{x}) = \frac{\partial}{\partial x^\nu}W_{\mu\nu}(\mathbf{x}) \qquad (13)$$

where $W_{\mu\nu}(\mathbf{x})$ is a stress tensor, which is responsible for the contribution of interparticle interaction to the force balance[21]

$$W_{\mu\nu}(\mathbf{x}) = -\frac{1}{2}\int d\mathbf{x}'\frac{x'^\mu x'^\nu}{|\mathbf{x}'|}\frac{\partial w(|\mathbf{x}'|)}{\partial|\mathbf{x}'|}$$
$$\times \int_0^1 \rho_2(\mathbf{x}+\lambda\mathbf{x}', \mathbf{x}-(1-\lambda)\mathbf{x}')d\lambda \quad (14)$$

In the next section we will show that parameter λ in Eq. (14) has a deep geometric meaning. It can be associated to the natural parameter for a geodesic (straight line in the present case) that connects two interacting particles (see also Appendix A).

Equations (11), (13) and (14) show that the net internal force, $F_\mu^{\text{kin}} + F_\mu^{\text{int}}$, is representable in a form of divergence of a symmetric second-rank tensor $\Pi_{\mu\nu}$. Tensor $\Pi_{\mu\nu}$ describes local internal stresses in the fluid. A contribution of the convective motion of particles to this tensor is known exactly[2]. It equals to the macroscopic momentum flow tensor, $mnv_\mu v_\nu$, where $\mathbf{v} = \mathbf{j}/n$ is the fluid's velocity. It is convenient to separate this contribution explicitly and rewrite the conservation laws of Eqs. (8), (10) in the following familiar form

$$D_t n + n\frac{\partial}{\partial x^\mu}v_\mu = 0, \qquad (15)$$

$$mnD_t v_\mu + \frac{\partial}{\partial x^\nu}P_{\mu\nu} + n\frac{\partial}{\partial x^\mu}U_{\text{ext}} = 0, \qquad (16)$$

where $D_t = \frac{\partial}{\partial t} + \mathbf{v}\nabla$ is the convective derivative and $P_{\mu\nu} = T_{\mu\nu} + W_{\mu\nu}$ is the exact stress tensor, which contains the kinetic, $T_{\mu\nu}$, and the interaction, $W_{\mu\nu}$, contributions. The interaction stress tensor, $W_{\mu\nu}$, is given by Eq. (14), while the kinetic part, $T_{\mu\nu}$, is defined as follows

$$T_{\mu\nu} = \frac{1}{2m}\langle(\widehat{q}_\mu\psi)^\dagger\widehat{q}_\nu\psi + (\widehat{q}_\nu\psi)^\dagger\widehat{q}_\mu\psi - \frac{1}{2}\delta_{\mu\nu}\nabla^2\widehat{n}\rangle \quad (17)$$

where $\widehat{\mathbf{q}} = -i\nabla - m\mathbf{v}$ is the operator of "relative" momentum which accounts for the above-mentioned separation of the macroscopic convective motion.

Equations (15) and (16) form a basis for a hydrodynamic description of a nonequilibrim many-body system. According to the Runge-Gross mapping theorem of TDDFT[22] the exact many-body wave function/density matrix (for given initial conditions) is a unique functional of velocity $\mathbf{v}(\mathbf{x},t)$. Therefore the stress tensor $P_{\mu\nu}$ is also a functional of $\mathbf{v}$. Hence Eqs. (15) and (16) can be viewed as a formally closed system of equations that completely determine the dynamics of collective variables $n(\mathbf{x},t)$ and $\mathbf{v}(\mathbf{x},t)$. This dynamics is governed by the external force, $n\partial_\mu U_{\text{ext}}$, and by the force of internal stress, $\partial_\nu P_{\mu\nu}$. Since the convective motion has been explicitly separated from the stress tensor, only the relative motion of particles contributes to $P_{\mu\nu}$. A particular form of $P_{\mu\nu}$ should be determined from the solution of a many-body problem in a reference frame moving with the "center-of-mass" velocity $\mathbf{v}(\mathbf{x},t)$. In the rest of the present paper we derive equations of many-body dynamics in this co-moving

frame and present simple illustrative examples of their solutions.

III. QUANTUM DYNAMICS IN THE LAGRANGIAN FRAME

A. Definition of the Lagrangian reference frame

Co-moving or Lagrangian frame is a local noninertial reference frame which moves with the velocity $\mathbf{v}(\mathbf{x}, t)$ of the fluid. Formally the transformation to the Lagrangian frame corresponds to a nonlinear change of variables $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}, t)$, which maps old coordinates $\mathbf{x}$ to new coordinates $\boldsymbol{\xi}$. For a given velocity distribution the transformation function, $\mathbf{x}(\boldsymbol{\xi}, t)$, is defined by the following initial value problem

$$\frac{\partial \mathbf{x}(\boldsymbol{\xi}, t)}{\partial t} = \mathbf{v}(\mathbf{x}(\boldsymbol{\xi}, t), t), \quad \mathbf{x}(\boldsymbol{\xi}, 0) = \boldsymbol{\xi} \qquad (18)$$

If the function $\mathbf{v}(\mathbf{x}, t)$ is continuous and satisfies the Lipschitz condition in $\mathbf{x}$, there exists a unique solution to the first order differential equation of Eq. (18)[23]. Therefore, under the above conditions on the velocity distribution, the map: $\mathbf{x} \to \boldsymbol{\xi}$ is unique and invertible.

Physically the function $\mathbf{x}(\boldsymbol{\xi}, t)$ corresponds to the trajectory of an infinitesimally small fluid element. Every fluid element (and therefore every trajectory) is uniquely labeled by the element's initial position – the Lagrangian coordinate $\boldsymbol{\xi}$. The inverse function $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x}, t)$, which determines the transformation from the Lagrangian to the laboratory reference frame, recovers the initial position of a fluid element that at instant t arrives at the point $\mathbf{x}$. The nonlinear transformation of coordinates, $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}, t)$,

induces a change of metric

$$(d\mathbf{x})^2 = g_{\mu\nu} d\xi^\mu d\xi^\nu, \qquad g_{\mu\nu} = \frac{\partial x^\alpha}{\partial \xi^\mu} \frac{\partial x^\alpha}{\partial \xi^\nu} \qquad (19)$$

In classical continuum mechanics the symmetric second rank tensor $g_{\mu\nu}(\boldsymbol{\xi}, t)$, Eq. (19), is known as Green's deformation tensor[1]. This tensor is normally used to characterize a deformed state of a system within the Lagrangian description. The corresponding contravariant tensor, $g^{\mu\nu}$, is defined as follows

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu, \qquad g^{\mu\nu} = \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial \xi^\nu}{\partial x^\alpha} \qquad (20)$$

Since the deformation tensor $g_{\mu\nu}$ has a meaning of the metric tensor in the Lagrangian $\boldsymbol{\xi}$-space, it should play a key role in the description of many-body dynamics. It is quite natural to expect that the general equations of motion in the Lagrangian frame should reduce to those in a space with time-dependent metric $g_{\mu\nu}(\boldsymbol{\xi}, t)$. Below we confirm this intuitive expectations by explicit calculations.

B. Equations of motion in a local noninertial reference frame

In this subsection we derive quantum equations of motion in a general noninertial (not necessarily Lagrangian) reference frame. The frame is defined by its velocity $\mathbf{v}(\mathbf{x}, t)$, which enters the trajectory equation of Eq. (18), and thus provides a unique and invertible map: $\mathbf{x} \to \boldsymbol{\xi}$. As a first step in the derivation we perform a nonlinear transformation of coordinates, $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}, t)$, in the equation of motion, Eq. (6), and in the commutation relations of Eq. (5). Straightforward calculations lead to the result

$$i\frac{\partial}{\partial t}\psi(\boldsymbol{\xi}) = \left[-\frac{1}{2m}\frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^\mu}\sqrt{g}g^{\mu\nu}\frac{\partial}{\partial \xi^\nu} + i\widetilde{v}^\mu(\boldsymbol{\xi}, t)\frac{\partial}{\partial \xi^\mu} + U_{\text{ext}}(\boldsymbol{\xi}, t) \right]\psi(\boldsymbol{\xi}) + \int d\boldsymbol{\xi}' w(l_{\boldsymbol{\xi},\boldsymbol{\xi}'})\psi^\dagger(\boldsymbol{\xi}')\psi(\boldsymbol{\xi}')\psi(\boldsymbol{\xi}) \qquad (21)$$

where $U_{\text{ext}}(\boldsymbol{\xi}, t) = U_{\text{ext}}(\mathbf{x}(\boldsymbol{\xi}, t), t)$, and field operators $\psi(\boldsymbol{\xi}, t)$ satisfy the following equal-time commutation relations

$$[\psi^\dagger(\boldsymbol{\xi}), \psi(\boldsymbol{\xi}')]_\pm = \frac{1}{\sqrt{g}}\delta(\boldsymbol{\xi} - \boldsymbol{\xi}'). \qquad (22)$$

To shorten the notations in Eq. (21) we omitted the explicit time dependence in the argument of ψ-operators. The first term in the brackets in Eq. (21) is the Laplace operator in a reference frame with metrics $g_{\mu\nu}$ (see, for example, Ref. 24), while the second term comes from the transformation of the time derivative in Eq. (6). This term is proportional to vector $\widetilde{v}^\mu(\boldsymbol{\xi}, t)$ that is the vector

of velocity, transformed to a new frame:

$$\widetilde{v}^\mu(\boldsymbol{\xi}, t) = \frac{\partial \xi^\mu}{\partial x^\nu} v^\nu(\mathbf{x}(\boldsymbol{\xi}, t), t). \qquad (23)$$

The interparticle distance, $l_{\boldsymbol{\xi},\boldsymbol{\xi}'}$, in the argument of the interaction potential in Eq. (21) equals to a length of geodesic that connects points $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$. Geodesic, $\mathbf{z}_{\boldsymbol{\xi},\boldsymbol{\xi}'}(\lambda)$, parameterized by a parameter λ ($0 < \lambda < 1$), is a solution to the following equation[24]

$$\ddot{z}^\mu(\lambda) + \Gamma^\mu_{\alpha\beta}(\mathbf{z})\dot{z}^\alpha(\lambda)\dot{z}^\beta(\lambda) = 0, \qquad (24)$$

where $\dot{\mathbf{z}} = \partial \mathbf{z}/\partial \lambda$, and $\Gamma^\mu_{\alpha\beta}$ is affine connection

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\frac{\partial g_{\nu\alpha}}{\partial \xi^\beta} + \frac{\partial g_{\nu\beta}}{\partial \xi^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial \xi^\nu} \right) \qquad (25)$$

Equation (24) should be solved with boundary conditions $\mathbf{z}(0) = \boldsymbol{\xi}$, $\mathbf{z}(1) = \boldsymbol{\xi}'$. It is convenient to parameterize geodesics by a natural parameter, which is chosen in such a way that an absolute value of the "velocity", $|\dot{\mathbf{z}}| = \sqrt{g_{\mu\nu}\dot{z}^\mu \dot{z}^\nu}$, becomes independent of λ along the curve $\mathbf{z}(\lambda)$. For this parameterization the length $l_{\boldsymbol{\xi},\boldsymbol{\xi}'}$, which enters Eq. (21), simply equals to $|\dot{\mathbf{z}}|$ at any point on the geodesic

$$l_{\boldsymbol{\xi},\boldsymbol{\xi}'} = \int_0^1 \sqrt{g_{\mu\nu}(\mathbf{z})\dot{z}^\mu(\lambda)\dot{z}^\nu(\lambda)}d\lambda = \sqrt{g_{\mu\nu}\dot{z}^\mu \dot{z}^\nu} \quad (26)$$

Equation (21) is the equation of motion for the operator $\psi(\boldsymbol{\xi},t) = \psi(\mathbf{x}(\boldsymbol{\xi},t),t)$. Due to the Jacobian factor $1/\sqrt{g}$ in the commutation relations of Eq. (22), the quantity $\psi(\boldsymbol{\xi},t)$ can not be interpreted as an operator for annihilation of a particle in a given point of $\boldsymbol{\xi}$-space. In particular, the operator $\widehat{n}(\boldsymbol{\xi}) = \psi^\dagger(\boldsymbol{\xi})\psi(\boldsymbol{\xi})$ does not correspond to the density operator in the new frame. By definition the density is a number of particles per unit volume that is changed under a volume non-preserving coordinate transformation. Therefore it is natural to define the physical field operators and the density operator as follows

$$\widetilde{\psi}(\boldsymbol{\xi}) = g^{\frac{1}{4}}\psi(\boldsymbol{\xi}), \quad \widetilde{\psi}^\dagger(\boldsymbol{\xi}) = g^{\frac{1}{4}}\psi^\dagger(\boldsymbol{\xi}), \quad (27)$$

$$\widehat{\widetilde{n}}(\boldsymbol{\xi}) = \widetilde{\psi}^\dagger(\boldsymbol{\xi})\widetilde{\psi}(\boldsymbol{\xi}) = \sqrt{g}\psi^\dagger(\boldsymbol{\xi})\psi(\boldsymbol{\xi}), \quad (28)$$

which automatically accounts for the proper change of a unit volume in the deformed reference frame. Obviously the redefined field operators $\widetilde{\psi}(\boldsymbol{\xi})$ satisfy the common commutations relations

$$[\widetilde{\psi}^\dagger(\boldsymbol{\xi}), \widetilde{\psi}(\boldsymbol{\xi}')]_\pm = \delta(\boldsymbol{\xi} - \boldsymbol{\xi}'). \quad (29)$$

The renormalization of ψ-operators, Eq. (27), is equivalent to the corresponding multiplicative redefinition of the many-body wave function. This redefinition is aimed to preserve the common probabilistic interpretation and the standard form of the normalization conditions in the new reference frame (similar arguments were suggested by Podolsky[25] in early days of quantum mechanics).

Let us show that the renormalization of field operators, Eq. (27), also simplifies the form of the equations of motion. First we note that the differential operator in the right hand side in Eq. (21) (first two terms in the square brackets) can be rearranged as follows

$$-\frac{1}{2m}\frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi^\mu}\sqrt{g}g^{\mu\nu}\frac{\partial}{\partial\xi^\nu} + i\widetilde{v}^\mu\frac{\partial}{\partial\xi^\mu}$$

$$= \frac{1}{\sqrt{g}}\frac{\widehat{K}_\mu\sqrt{g}\widehat{K}^\mu}{2m} - m\frac{\widetilde{v}_\mu\widetilde{v}^\mu}{2} - i\frac{1}{2\sqrt{g}}\left(\frac{\partial}{\partial\xi^\mu}\sqrt{g}\widetilde{v}^\mu\right) (30)$$

where we introduced an operator of "kinematic" momentum in the noninertial reference frame:

$$\widehat{K}_\mu = -i\frac{\partial}{\partial\xi^\mu} - m\widetilde{v}_\mu. \quad (31)$$

(Raising and lowering of tensor indices are performed according to the standard rules, i.e. $\widetilde{v}_\mu = g_{\mu\nu}\widetilde{v}^\nu$ or $\widehat{K}^\mu = g^{\mu\nu}\widehat{K}_\nu$.)

Using the equation of trajectory $\mathbf{x}(\boldsymbol{\xi},t)$, Eq. (18), and the definition of metric tensor $g_{\mu\nu}$, Eq. (19), one can prove the following identity

$$g^{-\frac{1}{4}}\frac{\partial g^{\frac{1}{4}}}{\partial t} = \frac{1}{4}\frac{\partial \ln g}{\partial t} = \frac{1}{2\sqrt{g}}\left(\frac{\partial}{\partial\xi^\mu}\sqrt{g}\widetilde{v}^\mu\right) \quad (32)$$

The quantity in the right hand side in Eq. (32) coincides with the last term in the right hand side in Eq. (30). Hence the sum of the corresponding term in the equation of motion, Eq. (21), and of the time derivative of ψ reduces to the following compact form

$$\frac{\partial\psi}{\partial t} + \frac{1}{2\sqrt{g}}\left(\frac{\partial}{\partial\xi^\mu}\sqrt{g}\widetilde{v}^\mu\right)\psi = g^{-\frac{1}{4}}\frac{\partial g^{\frac{1}{4}}\psi}{\partial t} = g^{-\frac{1}{4}}\frac{\partial\widetilde{\psi}}{\partial t} \quad (33)$$

Substituting Eq. (30) into Eq. (21) and using Eq. (33), we obtain the final equation of motion for the renormalized field operator $\widetilde{\psi}(\boldsymbol{\xi},t)$

$$i\frac{\partial\widetilde{\psi}(\boldsymbol{\xi})}{\partial t} = \left[g^{-\frac{1}{4}}\frac{\widehat{K}_\mu\sqrt{g}\widehat{K}^\mu}{2m}g^{-\frac{1}{4}} + U_{\text{ext}} - m\frac{\widetilde{v}_\mu\widetilde{v}^\mu}{2}\right]\widetilde{\psi}(\boldsymbol{\xi})$$

$$+ \int d\boldsymbol{\xi}' w(l_{\boldsymbol{\xi},\boldsymbol{\xi}'})\widetilde{\psi}^\dagger(\boldsymbol{\xi}')\widetilde{\psi}(\boldsymbol{\xi}')\widetilde{\psi}(\boldsymbol{\xi}) \quad (34)$$

Equation (34) allows us to recover a form of the transformed Hamiltonian $\widetilde{H}[\widetilde{\psi}^\dagger, \widetilde{\psi}]$, which, together with the commutation relations of Eq. (29), determines the dynamics of the system

$$\widetilde{H} = \widehat{\widetilde{T}} + \widehat{\widetilde{W}} + \widehat{\widetilde{U}}, \quad (35)$$

$$\widehat{\widetilde{T}} = \int d\boldsymbol{\xi}\sqrt{g}\left(\widehat{K}_\mu g^{-\frac{1}{4}}\widetilde{\psi}\right)^\dagger \frac{g^{\mu\nu}}{2m}\left(\widehat{K}_\nu g^{-\frac{1}{4}}\widetilde{\psi}\right), \quad (36)$$

$$\widehat{\widetilde{W}} = \frac{1}{2}\int d\boldsymbol{\xi}d\boldsymbol{\xi}' w(l_{\boldsymbol{\xi},\boldsymbol{\xi}'})\widetilde{\psi}^\dagger(\boldsymbol{\xi})\widetilde{\psi}^\dagger(\boldsymbol{\xi}')\widetilde{\psi}(\boldsymbol{\xi}')\widetilde{\psi}(\boldsymbol{\xi}), \quad (37)$$

$$\widehat{\widetilde{U}} = \int d\boldsymbol{\xi}\left(U_{\text{ext}} - m\frac{\widetilde{v}_\mu\widetilde{v}^\mu}{2}\right)\widetilde{\psi}^\dagger\widetilde{\psi} \quad (38)$$

Equations (34)–(38) represent the main results of this subsection. Equation (34) is the Heisenberg equation of motion for the physical field operator, while Eqs. (35)–(38) establish the rules for the transformation of the many-body Hamiltonian to an arbitrary local noninertial reference frame.

Formally the Hamiltonian of Eqs. (35)–(38) describes a system of quantum particles in the presence of an effective vector potential $m\widetilde{\mathbf{v}}(\boldsymbol{\xi},t)$ and an additional effective scalar potential $m\widetilde{\mathbf{v}}^2/2$. The particles live in a space with the time-dependent metric $g_{\mu\nu}(\boldsymbol{\xi},t)$ and interact via pairwise potential which depends on the length of geodesic connecting a pair of particles. Additional "potentials" and a nontrivial metric tensor are responsible for generalized inertia forces exerted on a particle in a

general noninertial reference frame. To get a transparent physical understanding of these forces it is instructive to look on dynamics in the semiclassical approximation. Since the most important inertial contributions enter only quadratic parts of the Hamiltonian (Eqs. (36) and (38)), we neglect for a moment the interaction, and consider an equation of motion for the Wigner function

$$\widetilde{f}_{\mathbf{p}}(\boldsymbol{\xi}, t) = \int e^{-i\mathbf{p}\boldsymbol{\eta}} \langle \widetilde{\psi}^\dagger(\boldsymbol{\xi} + \frac{\boldsymbol{\eta}}{2}, t)\widetilde{\psi}(\boldsymbol{\xi} - \frac{\boldsymbol{\eta}}{2}, t)\rangle d\boldsymbol{\xi}$$

in a gas of noninteracting particles. In the semiclassical limit the Wigner function satisfies the following kinetic equation

$$\frac{\partial \widetilde{f}_{\mathbf{p}}}{\partial t} + \frac{\partial \widetilde{H}(\mathbf{p}, \boldsymbol{\xi})}{\partial \mathbf{p}} \frac{\partial \widetilde{f}_{\mathbf{p}}}{\partial \boldsymbol{\xi}} - \frac{\partial \widetilde{H}(\mathbf{p}, \boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \frac{\partial \widetilde{f}_{\mathbf{p}}}{\partial \mathbf{p}} = 0, \qquad (39)$$

where $\widetilde{H}(\mathbf{p}, \boldsymbol{\xi})$ is the semiclassical Hamiltonian function, which corresponds the noninteracting part of Eq. (35):

$$\widetilde{H}(\mathbf{p}, \boldsymbol{\xi}) = \frac{g^{\mu\nu}}{2m}(p_\mu - m\widetilde{v}_\mu)(p_\nu - m\widetilde{v}_\nu)$$
$$+ U_{\text{ext}} - m\frac{\widetilde{v}_\mu \widetilde{v}^\mu}{2}. \qquad (40)$$

Substituting Eq. (40) into Eq. (39) we get the result

$$\frac{\partial \widetilde{f}_{\mathbf{p}}}{\partial t} + \frac{g^{\mu\nu}}{m}(p_\mu - m\widetilde{v}_\mu)\frac{\partial \widetilde{f}_{\mathbf{p}}}{\partial \xi^\nu}$$
$$- \left(\frac{\partial g^{\alpha\beta}}{\partial \xi^\nu}\frac{p_\alpha p_\beta}{2m} - \frac{\partial \widetilde{v}^\alpha}{\partial \xi^\nu}p_\alpha + \frac{\partial U_{\text{ext}}}{\partial \xi^\nu} \right)\frac{\partial \widetilde{f}_{\mathbf{p}}}{\partial p_\nu} = 0. \quad (41)$$

Inertia forces do not explicitly show up in Eq. (41). The reason is that Eq. (41) is the equation for the function $\widetilde{f}_{\mathbf{p}}$ which depends on the canonical momentum $\mathbf{p}$. The physical velocity of a particle in the new reference frame is proportional to the kinematic momentum $\mathbf{K} = \mathbf{p} - m\widetilde{\mathbf{v}}$ (i.e. $\partial \widetilde{H}/\partial p_\mu = K^\mu/m$). Therefore it is more natural physically to consider $\mathbf{K}$ as an independent variable in the kinetic equation. The distribution function of the kinematic momentum, $\widetilde{f}'_{\mathbf{K}}(\boldsymbol{\xi}, t)$, can be introduced as follows

$$\widetilde{f}'_{\mathbf{K}}(\boldsymbol{\xi}, t) = \widetilde{f}_{\mathbf{K}+m\widetilde{\mathbf{v}}}(\boldsymbol{\xi}, t). \qquad (42)$$

Performing the corresponding change of variables in Eq. (41) we obtain the final semiclassical equation of motion for the distribution function $\widetilde{f}'_{\mathbf{K}}(\boldsymbol{\xi}, t)$ in the local noninertial reference frame

$$\frac{\partial \widetilde{f}'_{\mathbf{K}}}{\partial t} + \frac{K^\nu}{m}\frac{\partial \widetilde{f}'_{\mathbf{K}}}{\partial \xi^\nu} - \Big[m\frac{\partial \widetilde{v}_\nu}{\partial t} + K^\mu \widetilde{F}_{\mu\nu} - \frac{\partial g_{\alpha\beta}}{\partial \xi^\nu}\frac{K^\alpha K^\beta}{2m}$$
$$+ \frac{\partial}{\partial \xi^\nu}\left(U_{\text{ext}} - m\frac{\widetilde{v}_\mu \widetilde{v}^\mu}{2}\right)\Big]\frac{\partial \widetilde{f}'_{\mathbf{K}}}{\partial K_\nu} = 0, \qquad (43)$$

where a skew-symmetric second rank tensor $\widetilde{F}_{\mu\nu}$ is defined as follows

$$\widetilde{F}_{\mu\nu} = \frac{\partial \widetilde{v}_\mu}{\partial \xi^\nu} - \frac{\partial \widetilde{v}_\nu}{\partial \xi^\mu}. \qquad (44)$$

Since tensor $\widetilde{F}_{\mu\nu}$ vanishes for an irrotational flow, we name it the vorticity tensor[26]. In the next section Eq. (43) will be applied to the derivation of generalized collisionless hydrodynamics[19,20].

The expression in the square brackets in Eq. (43) contains all inertia forces. These are all the terms except for the external force, $\partial_\nu U_{\text{ext}}$. The first term in the square brackets is the linear acceleration force, while the last term is related to the kinetic energy of a moving frame. In a particular case of a homogeneously rotating frame the last term is responsible for the centrifugal force. The second and the third terms in the square brackets correspond to inertia forces that depend on a velocity of a particular particle. The second term is the classical Coriolis force. This force is proportional to the skew-symmetric vorticity tensor, which defines a local angular velocity of the reference frame. The third, bilinear in particle's momentum term is less common. The corresponding inertia force makes a free particle to move along a geodesic in a local noninertial frame. Indeed, the third term in the square brackets in Eq. (43) can be rewritten as follows

$$\frac{1}{2m}\frac{\partial g_{\alpha\beta}}{\partial \xi^\nu}K^\alpha K^\beta = \frac{1}{m}g_{\nu\mu}\Gamma^\mu_{\alpha\beta}K^\alpha K^\beta, \qquad (45)$$

where we have used Eq. (25), which relates the affine connection $\Gamma^\mu_{\alpha\beta}$ to the metric tensor $g_{\mu\nu}$. The right hand side in Eq. (45) is easily recognized as a covariant component of the force in the equation of geodesic (see, for example, Eq. (24)).

C. Conserving quantities and balance equations

1. The continuity equation

The first problem we address in this subsection is a proper definition of the current operator, $\widehat{\widetilde{j}^\mu}$, in a general noninertial reference frame. The easiest way to establish a form of $\widehat{\widetilde{j}^\mu}$ is to derive the equation of motion for the density operator $\widehat{\widetilde{n}}(\boldsymbol{\xi}, t) = \widetilde{\psi}^\dagger(\boldsymbol{\xi}, t)\widetilde{\psi}(\boldsymbol{\xi}, t)$. Using Eq. (34) to compute the time derivative of the density operator we find that the desired equation indeed reduces to the common form of the continuity equation,

$$\frac{\partial \widehat{\widetilde{n}}}{\partial t} + \frac{\partial \widehat{\widetilde{j}^\mu}}{\partial \xi^\mu} = 0, \qquad (46)$$

if we define the current operator, $\widehat{\widetilde{j}^\mu}(\boldsymbol{\xi}, t)$, as follows

$$\widehat{\widetilde{j}^\mu} = g^{\mu\nu}\left[\frac{-i}{2m}\left(\widetilde{\psi}^\dagger \frac{\partial \widetilde{\psi}}{\partial \xi^\nu} - \frac{\partial \widetilde{\psi}^\dagger}{\partial \xi^\nu}\widetilde{\psi} \right) - \widetilde{v}_\nu \widetilde{\psi}^\dagger \widetilde{\psi} \right]. \qquad (47)$$

The standard form of the continuity equation, Eq. (46), should be considered as one more justification for the redefinition of field operators, Eq. (27). We would like to outline a very natural form of the current operator,

Eq. (47). Despite the presence of the Jacobian factors ($\sqrt{g}$ or $g^{1/4}$) in the Hamiltonian, they completely vanish in Eq. (47) (as well as in the definition of the density operator of Eq. (28)).

From this point we restrict ourselves to the Lagrangian frame, which is the local reference frame, moving with the velocity $\mathbf{v}$ of the fluid. In this special case the continuity equation admits a very simple solution. Let us calculate the expectation value of the current operator $\widehat{\widetilde{j}^{\mu}}$, Eq. (47). This can be done, for example, by transforming the right hand side in Eq. (47) back to the laboratory frame, and by using Eq. (9) together with the definition of the velocity, $\mathbf{v} = \mathbf{j}/n$. The result takes an extremely simple form

$$\widetilde{j}^{\mu}(\boldsymbol{\xi}, t) = \langle \widehat{\widetilde{j}^{\mu}}(\boldsymbol{\xi}, t) \rangle = 0. \tag{48}$$

Thus the current density is exactly zero in every point of the Lagrangian $\boldsymbol{\xi}$-space. This is of course not surprising, since an observer in the co-moving frame should not see any current. Combining Eq. (48) and the continuity equation of Eq. (46) we arrive at the conclusion that the density $\widetilde{n}(\boldsymbol{\xi}, t)$ is independent of time

$$\widetilde{n}(\boldsymbol{\xi}, t) = \widetilde{n}(\boldsymbol{\xi}, 0) = n_0(\boldsymbol{\xi}), \tag{49}$$

where $n_0(\mathbf{x})$ is the initial density distribution. Therefore in the Lagrangian frame not only the number of particles N is an integral of motion, but the density itself is also a conserving quantity. Evolution of the density in the laboratory frame can be calculated with the following formula (see Eq. (28))

$$n(\mathbf{x}, t) = \frac{\widetilde{n}(\boldsymbol{\xi}(\mathbf{x}, t), t)}{\sqrt{g(\boldsymbol{\xi}(\mathbf{x}, t), t)}} = \frac{n_0(\boldsymbol{\xi}(\mathbf{x}, t))}{\sqrt{g(\boldsymbol{\xi}(\mathbf{x}, t), t)}} \tag{50}$$

Equation (50) is, in fact, the explicit solution to the continuity equation of Eq. (8), which defines the density $n(\mathbf{x}, t)$ as a functional of velocity $\mathbf{v}(\mathbf{x}, t)$.

Equations (48) and (49) demonstrate the main advantage of the Lagrangian frame for the description of many-body dynamics. In this very special reference frame the inertia forces are adjusted to get exactly zero current density and therefore to keep the density of particles fixed during the whole evolution of the system. Equation (48) can be used to construct a complete many-body theory in the co-moving frame. The frame's velocity $\mathbf{v}$ enters the equation of motion, Eq. (34), as an external parameter. Imposing the local "gauge" condition of Eq. (48) we specify the reference frame and thus get the complete theory with all quantities defined by the initial conditions.

2. Local force balance in the Lagrangian frame

Let us turn to the local momentum conservation law. In the laboratory reference frame it is given by Eq. (16) (or, equivalently, by Eq. (10)). Since in the Lagrangian frame the current density is zero, the local momentum conservation law should reduce to the zero force condition – the inertia forces should exactly compensate the external force and the force of internal stresses. Below we derive an explicit form of this balance equation by the direct transformation of Eq. (16) to the Lagrangian coordinates $\boldsymbol{\xi}$. First we express the vector of velocity $\mathbf{v}$ and the stress tensor $P_{\mu\nu}$ in terms of the corresponding quantities, $\widetilde{\mathbf{v}}$ and $\widetilde{P}^{\mu}_{\nu}$, in the Lagrangian frame

$$v_{\mu} = \frac{\partial \xi^{\beta}}{\partial x^{\mu}} \widetilde{v}_{\beta}(\boldsymbol{\xi}, t), \tag{51}$$

$$P_{\mu\nu} = \frac{\partial \xi^{\beta}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial \xi^{\gamma}} \widetilde{P}^{\gamma}_{\beta}(\boldsymbol{\xi}, t). \tag{52}$$

Equation (51) follows the definition of $\widetilde{v}^{\mu}$, Eq. (23), while in Eq. (52) we adopted the standard tensor transformation rules[24]. Substituting Eqs. (51) and (52) into Eq. (16), transforming the derivatives, and multiplying the result by $\partial x^{\mu}/\partial \xi^{\alpha}$, we obtain the following equation

$$mn \frac{\partial x^{\mu}}{\partial \xi^{\alpha}} \frac{\partial}{\partial t} \left(\frac{\partial \xi^{\beta}}{\partial x^{\mu}} \widetilde{v}_{\beta} \right) + \frac{\partial x^{\mu}}{\partial \xi^{\alpha}} \frac{\partial \xi^{\delta}}{\partial x^{\nu}} \frac{\partial}{\partial \xi^{\delta}} \left(\frac{\partial \xi^{\beta}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial \xi^{\gamma}} \widetilde{P}^{\gamma}_{\beta} \right)$$
$$+ n \frac{\partial U_{\text{ext}}}{\partial \xi^{\alpha}} = 0 \tag{53}$$

The first term in Eq. (53) can be further simplified as follows

$$\frac{\partial x^{\mu}}{\partial \xi^{\alpha}} \frac{\partial}{\partial t} \frac{\partial \xi^{\beta}}{\partial x^{\mu}} \widetilde{v}_{\beta} = \frac{\partial \widetilde{v}_{\alpha}}{\partial t} - \widetilde{v}_{\beta} \frac{\partial \xi^{\beta}}{\partial x^{\mu}} \frac{\partial v^{\mu}}{\partial \xi^{\alpha}} = \frac{\partial \widetilde{v}_{\alpha}}{\partial t} - \frac{1}{2} \frac{\partial \widetilde{v}_{\beta} \widetilde{v}^{\beta}}{\partial \xi^{\alpha}} \tag{54}$$

In the sequence of transformations in Eq. (54) we used an obvious identity $\frac{\partial x^{\mu}}{\partial \xi^{\alpha}} \frac{\partial \xi^{\beta}}{\partial x^{\mu}} = \delta^{\beta}_{\alpha}$, the trajectory equation of Eq. (18), and the definition of $\widetilde{v}^{\mu}$, Eq. (23). A similar simplification of the second term in Eq. (53) is even more straightforward. One only needs to apply the chain rule for the calculation of the spatial derivative and take into account the following explicit representation for affine connection (see, for example, Ref. 24),

$$\Gamma^{\mu}_{\alpha\beta} = \frac{\partial \xi^{\mu}}{\partial x^{\gamma}} \frac{\partial^2 x^{\gamma}}{\partial \xi^{\alpha} \partial \xi^{\beta}}. \tag{55}$$

As a result we get a very natural expression for the second term in Eq. (53)

$$\frac{\partial x^{\mu}}{\partial \xi^{\alpha}} \frac{\partial \xi^{\delta}}{\partial x^{\nu}} \frac{\partial}{\partial \xi^{\delta}} \left(\frac{\partial \xi^{\beta}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial \xi^{\gamma}} \widetilde{P}^{\gamma}_{\beta} \right) = \widetilde{P}^{\beta}_{\alpha;\beta} \tag{56}$$

where the semicolon is used to denote the covariant derivative. The covariant divergence of the stress tensor in Eq. (56) is defined as follows[24,27]

$$\widetilde{P}^{\nu}_{\mu;\nu} = \frac{\partial \widetilde{P}^{\nu}_{\mu}}{\partial \xi^{\nu}} + \Gamma^{\nu}_{\nu\alpha} \widetilde{P}^{\alpha}_{\mu} - \Gamma^{\nu}_{\mu\alpha} \widetilde{P}^{\alpha}_{\nu}$$
$$= \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g} \widetilde{P}^{\nu}_{\mu}}{\partial \xi^{\nu}} - \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial \xi^{\mu}} \widetilde{P}^{\alpha\beta} \tag{57}$$

Substitution of Eqs. (54) and (56) into Eq. (53) leads to the final form of the force balance equation in the Lagrangian frame

$$\widetilde{n}\left[m\frac{\partial \widetilde{v}_\mu}{\partial t} + \frac{\partial}{\partial \xi^\mu}\left(U_{\text{ext}} - m\frac{\widetilde{v}_\nu \widetilde{v}^\nu}{2}\right)\right] + \sqrt{g}\widetilde{P}^\nu_{\mu;\nu} = 0. \quad (58)$$

A direct comparison of the force term in the kinetic equation of Eq. (43) and the term in the square brackets in Eq. (58) shows that the later is exactly the sum of the external force and two inertia forces that are independent of particle's momentum. These three forces are balanced by the force of internal stresses (the second term in Eq. (58)). The net force, exerted on every fluid element in the Lagrangian space, is zero, which results in a zero current density and a stationary particles' density distribution. It should be noted that the rest of inertia forces (those, which are different for different particles in a fluid element) implicitly present in the kinetic part of the stress tensor $\widetilde{P}_{\mu\nu}$. To see this more clearly we need to derive an explicit microscopic representation for this tensor.

3. Stress tensor in the Lagrangian frame

In general both kinetic, $\widetilde{T}_{\mu\nu}$, and interaction, $\widetilde{W}_{\mu\nu}$, contributions to the total stress tensor, $\widetilde{P}_{\mu\nu} = \widetilde{T}_{\mu\nu} + \widetilde{W}_{\mu\nu}$, can be found using the common transformations rules[24]

$$\widetilde{T}_{\mu\nu}(\boldsymbol{\xi},t) = \frac{\partial x^\alpha}{\partial \xi^\mu}\frac{\partial x^\beta}{\partial \xi^\nu}T_{\alpha\beta}(\mathbf{x}(\boldsymbol{\xi},t),t), \quad (59)$$

$$\widetilde{W}_{\mu\nu}(\boldsymbol{\xi},t) = \frac{\partial x^\alpha}{\partial \xi^\mu}\frac{\partial x^\beta}{\partial \xi^\nu}W_{\alpha\beta}(\mathbf{x}(\boldsymbol{\xi},t),t). \quad (60)$$

Here stress tensors, $T_{\alpha\beta}(\mathbf{x},t)$ and $W_{\alpha\beta}(\mathbf{x},t)$, in the laboratory reference frame are given by Eqs. (17) and (14) respectively. We shall however follow another route, which takes full advantage of geometric ideas we develop in this paper. The transformed many-body Hamiltonian

of Eqs. (35)–(38), is an explicit functional of the metric tensor $g_{\mu\nu}$. Therefore we can find the required stress tensor by computing the variational derivative of the energy with respect to the metric tensor. More precisely, we make use of the fact that under a small variation of the metric, the variations of the kinetic energy, Eq. (36), and of the energy of interparticle interaction, Eq. (37), are related to tensors $\widetilde{T}_{\mu\nu}$ and $\widetilde{W}_{\mu\nu}$ respectively

$$\delta\langle\widehat{\widetilde{T}}\rangle = \int d\boldsymbol{\xi}\frac{\sqrt{g}}{2}\delta g^{\mu\nu}\widetilde{T}_{\mu\nu} = -\int d\boldsymbol{\xi}\frac{\sqrt{g}}{2}\delta g_{\mu\nu}\widetilde{T}^{\mu\nu} \quad (61)$$

$$\delta\langle\widehat{\widetilde{W}}\rangle = \int d\boldsymbol{\xi}\frac{\sqrt{g}}{2}\delta g^{\mu\nu}\widetilde{W}_{\mu\nu} = -\int d\boldsymbol{\xi}\frac{\sqrt{g}}{2}\delta g_{\mu\nu}\widetilde{W}^{\mu\nu} \quad (62)$$

Such a definition of the stress tensors is closely related to the common definition of the energy-momentum tensor in general relativity (see, for example, Ref. 27). The main advantage of this definition is that it automatically gives a symmetric form of the stress/energy-momentum tensors. Recently a very similar approach has been used to derive a microscopic expression for the stress tensor in the equilibrium quantum many-body system within the local density approximation[13]. The paper of Ref. 13 also contains a general discussion of the above geometric definition of the stress tensors in the context of nonrelativistic quantum mechanics.

The variation of the Hamiltonian should be taken at constant $\widetilde{\psi}$-variables (since they satisfy the equations of motion)[13,27] and at constant velocity $\widetilde{\mathbf{v}}$. The kinetic energy operator $\widehat{\widetilde{T}}$ of (36) contains the metric tensor only in a form of $g^{\mu\nu}$, $\sqrt{g}$ and $g^{-\frac{1}{4}}$. Noting that

$$\delta\sqrt{g} = -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu}, \qquad \delta g^{-\frac{1}{4}} = \frac{1}{4}g^{-\frac{1}{4}}g_{\mu\nu}\delta g^{\mu\nu},$$

we can easily compute the variation of $\widehat{\widetilde{T}}$ and then identify the kinetic stress tensor using Eq. (61). The result of the calculations takes the form

$$\widetilde{T}_{\mu\nu}(\boldsymbol{\xi},t) = \frac{1}{2m}\left\langle\left(\widehat{K}_\mu g^{-\frac{1}{4}}\widetilde{\psi}\right)^\dagger\left(\widehat{K}_\nu g^{-\frac{1}{4}}\widetilde{\psi}\right) + \left(\widehat{K}_\nu g^{-\frac{1}{4}}\widetilde{\psi}\right)^\dagger\left(\widehat{K}_\mu g^{-\frac{1}{4}}\widetilde{\psi}\right) - \frac{1}{2}g_{\mu\nu}\frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^\alpha}\sqrt{g}g^{\alpha\beta}\frac{\partial}{\partial \xi^\beta}\frac{\widetilde{\psi}^\dagger\widetilde{\psi}}{\sqrt{g}}\right\rangle \quad (63)$$

If we evaluate the right hand side in Eq. (63) for Euclidian metric, $g_{\mu\nu} = \delta_{\mu\nu}$, we immediately recover Eq. (17). Therefore the commonly used symmetric form of the kinetic stress tensor $T_{\mu\nu}$, Eq. (17), is in exact correspondence with the geometric definition of Eq. (61).

Calculation of the variation $\delta\langle\widehat{\widetilde{W}}\rangle$, Eq. (62), is a little bit more involved. Since the interaction Hamiltonian of Eq. (37) depends on $g_{\mu\nu}$ only via the length of geodesic,

we have

$$\delta\langle\widehat{\widetilde{W}}\rangle = \frac{1}{2}\int d\boldsymbol{\eta}d\boldsymbol{\eta}'\delta l_{\boldsymbol{\eta},\boldsymbol{\eta}'}\frac{\partial w(l_{\boldsymbol{\eta},\boldsymbol{\eta}'})}{\partial l_{\boldsymbol{\eta},\boldsymbol{\eta}'}}\widetilde{\rho}_2(\boldsymbol{\eta},\boldsymbol{\eta}') \quad (64)$$

where $\widetilde{\rho}_2(\boldsymbol{\eta},\boldsymbol{\eta}') = \langle\widetilde{\psi}^\dagger(\boldsymbol{\eta})\widetilde{\psi}^\dagger(\boldsymbol{\eta}')\widetilde{\psi}(\boldsymbol{\eta}')\widetilde{\psi}(\boldsymbol{\eta})\rangle$. The next step is to compute the variation of the functional $l_{\boldsymbol{\eta},\boldsymbol{\eta}'}[g_{\mu\nu}]$, Eq. (26). Let λ in Eq. (26) be a natural parameter for a geodesic in the space with "unperturbed"

metric $g_{\mu\nu}$ (not the "full" metric $g_{\mu\nu} + \delta g_{\mu\nu}$). For this parameterization the variation of $l_{\boldsymbol{\eta},\boldsymbol{\eta}'}[g_{\mu\nu}]$ takes the form

$$\delta l_{\boldsymbol{\eta},\boldsymbol{\eta}'} = \frac{1}{2l_{\boldsymbol{\eta},\boldsymbol{\eta}'}} \int_0^1 d\lambda \delta g_{\mu\nu}(\mathbf{z}(\lambda))\dot{z}^\mu(\lambda)\dot{z}^\nu(\lambda)$$

$$= \int d\boldsymbol{\xi} \int_0^1 d\lambda \delta(\boldsymbol{\xi} - \mathbf{z}(\lambda))\delta g_{\mu\nu}(\boldsymbol{\xi})\frac{\dot{z}^\mu(\lambda)\dot{z}^\nu(\lambda)}{2l_{\boldsymbol{\eta},\boldsymbol{\eta}'}} \quad (65)$$

where $\mathbf{z}(\lambda) = \mathbf{z}_{\boldsymbol{\eta},\boldsymbol{\eta}'}(\lambda)$ is the geodesic which connects points $\boldsymbol{\eta}$ and $\boldsymbol{\eta}'$. Substituting Eq. (65) into Eq. (64), and using the definition of Eq. (62) we get the following representation for the interaction part of the stress tensor

$$\widetilde{W}^{\mu\nu}(\boldsymbol{\xi},t) = -\frac{1}{2\sqrt{g}} \int_0^1 d\lambda \int d\boldsymbol{\eta} d\boldsymbol{\eta}' \delta(\boldsymbol{\xi} - \mathbf{z}_{\boldsymbol{\eta},\boldsymbol{\eta}'}(\lambda))\frac{\dot{z}^\mu_{\boldsymbol{\eta},\boldsymbol{\eta}'}(\lambda)\dot{z}^\nu_{\boldsymbol{\eta},\boldsymbol{\eta}'}(\lambda)}{l_{\boldsymbol{\eta},\boldsymbol{\eta}'}}\frac{\partial w(l_{\boldsymbol{\eta},\boldsymbol{\eta}'})}{\partial l_{\boldsymbol{\eta},\boldsymbol{\eta}'}}\widetilde{\rho}_2(\boldsymbol{\eta},\boldsymbol{\eta}') \qquad (66)$$

Let us evaluate the right hand side in Eq. (66) at Euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$. In this case $l_{\boldsymbol{\eta},\boldsymbol{\eta}'} = |\boldsymbol{\eta} - \boldsymbol{\eta}'|$ while the geodesic (parameterized by the natural parameter) is a straight line

$$\mathbf{z}_{\boldsymbol{\eta},\boldsymbol{\eta}'}(\lambda) = \boldsymbol{\eta} + (\boldsymbol{\eta}' - \boldsymbol{\eta})\lambda$$

The above expressions for $l_{\boldsymbol{\eta},\boldsymbol{\eta}'}$ and $\mathbf{z}_{\boldsymbol{\eta},\boldsymbol{\eta}'}(\lambda)$ should be substituted into Eq. (66). Introducing a new variable $\boldsymbol{\xi}' = \boldsymbol{\eta}' - \boldsymbol{\eta}$, and removing the delta-function by the integration over $\boldsymbol{\eta}$ we obtain at the result that exactly coincides with Eq. (14). Therefore the symmetric representation of Eq. (14), which has been obtained in the previous section by somewhat artificial manipulations, has a clear geometric meaning. In particular, the internal parameter λ in Eq. (14) is the natural parameter for a geodesic connecting two interacting particles.

Equations (63) and (66) are the principal results of the present subsection. They define explicit microscopic representations for the stress tensors in a local noninertial reference frame.

The zero force condition of Eq. (58) with $\widetilde{P}_{\mu\nu}$ of Eqs. (63) and (66) is equivalent to the requirement of zero current density, Eq. (48). Hence we can use Eq. (58) as an alternative "gauge" condition to fix the velocity parameter $\widetilde{\mathbf{v}}(\boldsymbol{\xi},t)$, entering many-body equations of motion, Eq. (34).

IV. EXAMPLES AND APPLICATIONS

A. The harmonic potential theorem

As a first simple example of application of our general formalism, we consider a many-body dynamics in the presence of the following external potential

$$U_{\text{ext}}(\mathbf{x},t) = \frac{1}{2}m\omega_{\mu\nu}x^\mu x^\nu + E_\mu(t)x^\mu \qquad (67)$$

where $\omega_{\mu\nu}$ is a constant tensor and $E_\mu(t)$ is a time-dependent vector (without loss of generality we can put

$E_\mu(0) = 0$). The initial value problem with the external potential of Eq. (67) is exactly solvable, which is known as the harmonic potential theorem (HPT)[6]. It is also known that HPT is related to the covariance of the time-dependent Schödinger equation under the transformation to a global accelerated reference frame[7,8]. Therefore our formulation of the many-body problem should be perfectly suited to the demonstration of HPT.

Within the present Lagrangian formulation one needs to find a selfconsistent solution to the many-body equation of motion, Eq. (34), and to the force balance equation, Eq. (58). Let us assume that velocity $\mathbf{v}(\mathbf{x},t)$, which defines (via Eq. (18)) the motion of the reference frame, is a function of t only: $\mathbf{v}(\mathbf{x},t) = \mathbf{V}(t)$. In this case the trajectory of a fluid element takes a form

$$\mathbf{x}(\boldsymbol{\xi},t) = \boldsymbol{\xi} + \mathbf{R}(t), \qquad (68)$$

where $\mathbf{R}(t)$ is a solution to the following Cauchy problem

$$\frac{\partial\mathbf{R}(t)}{\partial t} = \mathbf{V}(t), \qquad \mathbf{R}(0) = 0$$

Clearly, if our anzatz, $\mathbf{v}(\mathbf{x},t) = \mathbf{V}(t)$, is a selfconsistent solution, then $\mathbf{R}(t)$ should correspond to the center-of-mass coordinate. Using Eq. (68) we get the following results for the metric tensor $g_{\mu\nu}$, the velocity $\widetilde{\mathbf{v}}(\boldsymbol{\xi},t)$, and the effective potential, which enter Eqs. (34) and (58)

$$g_{\mu\nu} = \delta_{\mu\nu}, \qquad \widetilde{\mathbf{v}}(\boldsymbol{\xi},t) = \mathbf{V}(t) \qquad (69)$$

$$U_{\text{ext}}(\mathbf{x}(\boldsymbol{\xi},t),t) - m\frac{\widetilde{v}_\nu\widetilde{v}^\nu}{2} = \frac{1}{2}m\omega_{\mu\nu}\xi^\mu\xi^\nu - L(t)$$
$$+ \xi^\mu[m\omega_{\mu\nu}R^\nu(t) + E_\mu(t)] \qquad (70)$$

Function $L(t)$ in Eq. (70) is the classical Lagrangian for a particle moving in the harmonic potential of Eq. (67)

$$L(t) = \frac{m}{2}\mathbf{V}^2(t) - \frac{1}{2}m\omega_{\mu\nu}R^\mu(t)R^\nu(t) - E_\mu(t)R^\mu(t)$$

The equation of motion, Eq. (34), and the force balance equation, Eq. (58), simplify respectively as follows

$$i\frac{\partial \widetilde{\psi}'}{\partial t} = \left(-\frac{\nabla_{\boldsymbol{\xi}}^2}{2m} + \frac{1}{2}m\omega_{\mu\nu}\xi^\mu\xi^\nu \right)\widetilde{\psi}' + \int d\boldsymbol{\xi}' w(|\boldsymbol{\xi} - \boldsymbol{\xi}'|)\widehat{\widetilde{n}}(\boldsymbol{\xi}')\widetilde{\psi}'(\boldsymbol{\xi}) + \xi^\mu \left[m\frac{\partial V_\mu(t)}{\partial t} + m\omega_{\mu\nu}R^\nu(t) + E_\mu(t) \right]\widetilde{\psi}' \quad (71)$$

$$m\frac{\partial V_\mu(t)}{\partial t} + m\omega_{\mu\nu}R^\nu(t) + E_\mu(t) + \left[\frac{1}{n_0(\boldsymbol{\xi})}\frac{\partial}{\partial \xi^\nu}\widetilde{P}_{\mu\nu}(\boldsymbol{\xi}, t) + m\omega_{\mu\nu}\xi^\nu \right] = 0 \quad (72)$$

where we also performed a gauge transformation, $\widetilde{\psi}'(\boldsymbol{\xi}, t) = \widetilde{\psi}(\boldsymbol{\xi}, t)\exp[-im\mathbf{V}\boldsymbol{\xi} + i\int_0^t Ldt']$, which corresponds to the transformation from the canonical to the kinematic momentum.

Let initially the system is prepared in a stationary state (or in arbitrary mixture of stationary states). This means that at $t = 0$ the stationary force balance equation is fulfilled

$$\frac{1}{n_0(\boldsymbol{\xi})}\frac{\partial}{\partial \xi^\nu}\widetilde{P}_{\mu\nu}(\boldsymbol{\xi}, 0) + m\omega_{\mu\nu}\xi^\nu = 0. \quad (73)$$

If at all $t > 0$ the center-of-mass coordinate $\mathbf{R}(t)$ satisfies the classical equation of motion

$$m\frac{\partial^2 R_\mu(t)}{\partial t^2} + m\omega_{\mu\nu}R^\nu(t) + E_\mu(t) = 0, \quad (74)$$

then both the equation of motion, Eq. (71), and the force balance equation, Eq. (72), preserve their initial (stationary) form. Therefore the many-body system in the co-moving frame remains in the initial stationary state, in particular, $\widetilde{P}_{\mu\nu}(\boldsymbol{\xi}, t) = \widetilde{P}_{\mu\nu}(\boldsymbol{\xi}, 0)$. This statement is the essence of HPT[6]. Within the present formulation of many-body dynamics it appears quite naturally. In fact, HPT is a built-in property of our selfconsistent approach. This actually means that any approximate treatment of a selfconsistent system of Eqs. (34), (58) should automatically satisfy HPT.

B. Geometric formulation of generalized hydrodynamics: nonlinear elasticity of a collisionless Fermi gas

The HPT type of motion provides an extremely simple example of many-body dynamics without any deformation of local fluid elements ($g_{\mu\nu}^{\text{HPT}} = \delta_{\mu\nu}$). In this subsection we apply our approach to a much more general situation with a nontrivial dynamics of a fluid. Namely, we consider a semiclassical dynamics of an interacting Fermi system in the time-dependent Hartree approximation. The problem reduces to the selfconsistent solution of a semiclassical collisionless kinetic equation (see Eq. (43))

$$\frac{\partial \widetilde{f}'_{\mathbf{K}}}{\partial t} + \frac{K^\nu}{m}\frac{\partial \widetilde{f}'_{\mathbf{K}}}{\partial \xi^\nu} - \left[m\frac{\partial \widetilde{v}_\nu}{\partial t} + K^\mu\widetilde{F}_{\mu\nu} - \frac{\partial g_{\alpha\beta}}{\partial \xi^\nu}\frac{K^\alpha K^\beta}{2m} \right.$$
$$\left. + \frac{\partial}{\partial \xi^\nu}\left(U - m\frac{\widetilde{v}_\mu\widetilde{v}^\mu}{2} \right) \right]\frac{\partial \widetilde{f}'_{\mathbf{K}}}{\partial K_\nu} = 0, \quad (75)$$

and a force balance equation (see Eq. (58))

$$m\frac{\partial \widetilde{v}_\mu}{\partial t} + \frac{\partial}{\partial \xi^\mu}\left(U - m\frac{\widetilde{v}_\nu\widetilde{v}^\nu}{2} \right) + \frac{\sqrt{g}}{n_0}\widetilde{P}^\nu_{\mu;\nu} = 0. \quad (76)$$

Here $U = U_{\text{ext}}(\boldsymbol{\xi}, t) + U_{\text{H}}(\boldsymbol{\xi}, t)$ is a sum of the external potential and the Hartree potential,

$$U_{\text{H}}(\boldsymbol{\xi}, t) = \int w(l_{\boldsymbol{\xi}, \boldsymbol{\xi}'})n_0(\boldsymbol{\xi}')d\boldsymbol{\xi}'.$$

Since the interaction effects are already included (on the mean field level) in the selfconsistent potential, only kinetic part of the stress tensor contributes to Eq. (76), i.e. $\widetilde{P}_{\mu\nu} = m^{-1}\sum_{\mathbf{K}}K_\mu K_\nu \widetilde{f}'_{\mathbf{K}}/\sqrt{g}$. The last expression for $\widetilde{P}_{\mu\nu}$ is a plain semiclassical limit of the general kinetic stress tensor, Eq. (63).

The problem of solving Eqs. (75), (76) can be reformulated as follows. Let us substitute the sum of the inertia and the external forces from the balance equation, Eq. (76), into the kinetic equation of Eq. (75). After the substitution the potential U in Eq. (75) cancels out and the kinetic equation reduces to the following universal form

$$\frac{\partial \widetilde{f}'_{\mathbf{K}}}{\partial t} + \frac{K^\nu}{m}\frac{\partial \widetilde{f}'_{\mathbf{K}}}{\partial \xi^\nu} - \left[K^\mu\widetilde{F}_{\mu\nu} \right.$$
$$\left. - \frac{\partial g_{\alpha\beta}}{\partial \xi^\nu}\frac{K^\alpha K^\beta}{2m} - \frac{\sqrt{g}}{n_0}\widetilde{P}^\mu_{\nu;\mu} \right]\frac{\partial \widetilde{f}'_{\mathbf{K}}}{\partial K_\nu} = 0 \quad (77)$$

where the stress tensor $\widetilde{P}_{\mu\nu}$ is defined as follows

$$\widetilde{P}_{\mu\nu}(\boldsymbol{\xi}, t) = \frac{1}{\sqrt{g}}\sum_{\mathbf{K}}\frac{K_\mu K_\nu}{m}\widetilde{f}'_{\mathbf{K}}(\boldsymbol{\xi}, t) \quad (78)$$

Equations (77) and (78) constitute a closed set, which is structurally similar to the system of Vlasov equations with a selfconsistent force. The skew-symmetric vorticity tensor, $\widetilde{F}_{\mu\nu}(\boldsymbol{\xi}, t)$, and the symmetric deformation tensor, $g_{\mu\nu}(\boldsymbol{\xi}, t)$, enter Eqs. (77) and (78) as external parameters, which govern the evolution of the system. Hence these equations define a distribution function, $\widetilde{f}'_{\mathbf{K}}(\boldsymbol{\xi}, t)$, as a unique functional of $\widetilde{F}_{\mu\nu}$ and $g_{\mu\nu}$, provided the initial condition, $\widetilde{f}'_{\mathbf{K}}(\boldsymbol{\xi}, 0) = \widetilde{f}^{(0)}_{\mathbf{K}}(\boldsymbol{\xi})$, is given. Equation (78) determines the stress tensor as a universal (i. e. independent of external potential) functional of $\widetilde{F}_{\mu\nu}$ and $g_{\mu\nu}$:

$$\widetilde{P}_{\mu\nu} = \widetilde{P}_{\mu\nu}[\widetilde{F}_{\mu\nu}, g_{\mu\nu}](\boldsymbol{\xi}, t). \quad (79)$$

The vorticity and the deformation tensors contain nine independent scalar functions (three from $\widetilde{F}_{\mu\nu}$ and six from $g_{\mu\nu}$) which completely describe a deformed state of a system[1]. Hence Eq. (79) plays a role of a generalized "equation of state" which relates the stress tensor to the deformation. It is worth mentioning that the existence of such an equation of state is a direct consequence of Runge-Gross mapping theorem[22] in TDDFT.

Substituting the functional of Eq. (79) into Eq. (76) we obtain a hydrodynamic equation of motion which determines the evolution of velocity for a given external potential. Therefore the description of many-body dynamics consists of two separate problems. The first one corresponds to the universal kinetic problem of Eqs. (77), (78). By solving these equations we find the stress tensor functional, Eq. (79) (the generalized equation of state). The second problem is to compute the velocity and density distributions by solving the closed set of hydrodynamics equations, Eqs. (18) and (76).

The universal kinetic problem of Eqs. (77), (78) can be solved explicitly in the case of a fast long wavelength dynamics, i. e. if the deformation tensor is a fast function of time, but slowly changes in space. More precisely, we assume that the characteristic length scale, L, of the deformation inhomogeneity is much large than τu, where τ is the time scale of a dynamical process and u is the characteristic velocity of a particle. This situation is, for example, common in Coulomb systems where the plasma frequency determines the characteristic time scale of dynamics, while the corresponding spatial variations of the density can be arbitrary slow. Let us estimate different terms in Eq. (77) under the above assumption. The first and the second terms in the right hand side in Eq. (77) are of the order of $1/\tau$ and u/L respectively. Both terms in the second line in Eq. (77) also give a contribution $\sim u/L$, while the term, related to the Coriolis force, is proportional to $\widetilde{F} \sim \widetilde{v}_T/L$, where $\widetilde{v}_T$ is a rotational (or transverse) component of the velocity. According to the force balance equation of Eq. (76), for any physical velocity the transverse part of the linear acceleration is compensated by the transverse part of the vector $\frac{\sqrt{g}}{n_0}\widetilde{P}^\nu_{\mu;\nu}$. Hence $\widetilde{v}_T$ should be proportional to $\tau u^2/L$. This means that the contribution of Coriolis force to the kinetic equation is of the order of $\tau u^2/L^2$. Therefore, to the leading order in the small parameter $\gamma = \tau u/L \ll 1$, only the first term in Eq. (77) gives a nonvanishing contribution. Thus the universal problem of Eqs. (77), (78) reduces to the following trivial equation

$$\frac{\partial}{\partial t}\widetilde{f}'_{\mathbf{K}}(\boldsymbol{\xi}, t) = 0. \qquad (80)$$

Equation (80) shows that for a fast, small-gradient evolution the distribution function in the Lagrangian frame preserves its initial form: $\widetilde{f}'_{\mathbf{K}}(\boldsymbol{\xi}, t) = \widetilde{f}^{(0)}_{\mathbf{K}}(\boldsymbol{\xi})$. In this respect the dynamics reminds the HPT type of motion. However the evolution of the velocity is by far not trivial. Below we consider a system which evolves from the equilibrium state. Substituting the equilibrium distribution

function into Eq. (78) we get the stress tensor functional

$$\widetilde{P}_{\mu\nu}(\boldsymbol{\xi}, t) = \frac{\delta_{\mu\nu}}{\sqrt{g(\boldsymbol{\xi}, t)}}P_0(\boldsymbol{\xi}), \qquad (81)$$

which is proportional to the initial equilibrium pressure, $P_0(\boldsymbol{\xi})$. The last step is to substitute the nonadiabatic "equation of state", Eq. (81), into the force balance equation of Eq. (76). This results in the following "hydrodynamic" equation of motion

$$mn_0\frac{\partial \widetilde{v}_\mu}{\partial t} + n_0\frac{\partial}{\partial \xi^\mu}\left(U - m\frac{\widetilde{v}_\nu\widetilde{v}^\nu}{2}\right)$$
$$+ \frac{\partial g^{\mu\nu}P_0}{\partial \xi^\nu} + \frac{1}{2}P_0\frac{\partial g^{\alpha\alpha}}{\partial \xi^\mu} = 0, \qquad (82)$$

where we used the definition of the covariant divergence, Eq. (57), to compute the stress force, $\sqrt{g}\widetilde{P}^\nu_{\mu;\nu}$, in Eq. (76). We would like to outline that $n_0(\boldsymbol{\xi})$ and $P_0(\boldsymbol{\xi})$ in Eq. (82) are the time independent initial density and pressure respectively. Equations (82) and (18) constitute a closed set of continuum mechanics equations which describe a long wavelength dynamics of a Fermi gas in the time-dependent Hartree approximation. Since the stress force in Eq. (82) depends only on the deformation tensor, $g_{\mu\nu}$, it is natural to interpret Eqs. (82) and (18) as a nonlinear elasticity theory of a Fermi gas. In the case of small deformations this theory reduces to the standard linear elasticity theory with a nonzero shear modulus. Indeed, in the linear regime Eq. (18) takes the form

$$\frac{\partial \mathbf{u}(\boldsymbol{\xi}, t)}{\partial t} = \mathbf{v}(\boldsymbol{\xi}, t), \qquad (83)$$

where $\mathbf{u} = \mathbf{x} - \boldsymbol{\xi}$ is the displacement vector. The deformation tensor reduces to the common linearized expression

$$g^{\mu\nu} = \delta_{\mu\nu} - \frac{\partial u_\mu}{\partial \xi^\nu} - \frac{\partial u_\nu}{\partial \xi^\mu} \qquad (84)$$

Assuming for simplicity that the unperturbed state is homogeneous, and substituting Eqs. (83) and (84) into Eq. (82), we get the following equation of motion for the displacement vector

$$mn_0\frac{\partial^2 u_\mu}{\partial t^2} - \frac{\partial \sigma_{\mu\nu}}{\partial \xi^\nu} + n_0\frac{\partial \delta U}{\partial \xi^\mu} = 0. \qquad (85)$$

The linearized stress tensor $\sigma_{\mu\nu}$ takes the standard elastic form

$$\sigma_{\mu\nu} = \delta_{\mu\nu}K\frac{\partial u_\alpha}{\partial \xi^\alpha} + \mu\left(\frac{\partial u_\mu}{\partial \xi^\nu} - \frac{\partial u_\nu}{\partial \xi^\mu} - \delta_{\mu\nu}\frac{2}{3}\frac{\partial u_\alpha}{\partial \xi^\alpha}\right), \quad (86)$$

where $K = \frac{5}{3}P_0$ and $\mu = P_0$ are the bulk modulus and the shear modulus of a Fermi gas respectively[19,20,28].

Full nonlinear set of Eqs. (82), (18) is equivalent to the generalized collisionless hydrodynamics derived in Refs. 19,20 (see also Ref. 29). In fact, Eqs. (82), (18) and the generalized hydrodynamics of Refs. 19,20 correspond to the same theory in the Lagrangian and Eulerian

formulations respectively. An advantage of the present Lagrangian formulation is the explicit form of the stress tensor $\widetilde{P}_{\mu\nu}$, Eq. (81). The Lagrangian point of view also gives a very clear microscopic picture of a fast collisionless dynamics. This is a kind of evolution of a many-body system with almost time-independent distribution of particles inside every moving and deforming fluid element. Using the nonadiabatic equation of state in the Lagrangian frame, Eq. (81), we can easily recover the corresponding expression for the stress tensor $P_{\mu\nu}(\mathbf{x}, t)$ in the laboratory frame

$$P_{\mu\nu}(\mathbf{x}, t) = \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}} \widetilde{P}_{\alpha\beta}(\xi(\mathbf{x}, t), t)$$
$$= \bar{g}_{\mu\nu}(\mathbf{x}, t) \sqrt{\bar{g}(\mathbf{x}, t)} P_0(\xi(\mathbf{x}, t)) \qquad (87)$$

Here $\bar{g}_{\mu\nu}(\mathbf{x}, t)$ is Cauchy's deformation tensor[1]

$$\bar{g}_{\mu\nu}(\mathbf{x}, t) = \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\alpha}}{\partial x^{\nu}}, \qquad \bar{g} = 1/g \qquad (88)$$

One can check that $P_{\mu\nu}(\mathbf{x}, t)$ of Eq. (87) is a solution to the equation for the stress tensor derived in Refs. 19,20. The stress tensor $P_{\mu\nu}(\mathbf{x}, t)$, Eq. (87), enters the force balance equation in the laboratory frame, Eq. (16). This is a hydrodynamic equation of motion in Eulerian description. It is quite natural that $P_{\mu\nu}(\mathbf{x}, t)$ depends on $\bar{g}_{\mu\nu}(\mathbf{x}, t)$ since Cauchy's tensor is a common characteristics of deformations in Eulerian picture.

In contrast to the stress tensor in the Lagrangian frame, Eq. (81), the stress tensor of Eq. (87) is a highly nonlocal function. It is proportional to the pressure P_0 at the initial position of a fluid element which is currently at $\mathbf{x}$ ($\mathbf{x}$ is an independent variable in Eq. (16)). The locality of the stress force in Eq. (82) is a key property of the present Lagrangian formulation of a nonadiabatic continuum mechanics of a Fermi gas. This formulations should be much more convenient for applications to particular nonlinear problems.

V. CONCLUSION

We applied the idea of the Lagrangian description in continuum mechanics to the theory of nonequilibrium quantum many-body systems. Reformulation of the microscopic many-body theory in terms of Lagrangian coordinates corresponds to the transformation to the local noninertial reference frame moving with the flow (the comoving Lagrangian frame). This transformation allows to separate the convective motion of particles, which is a direct generalization of the common separation of the center-of-mass motion in homogeneous systems. The motion of particles in the Lagrangian frame is influenced by the external forces and by generalized inertia forces. We have shown that the inertia forces can be described in purely geometric terms of Green's deformation tensor $g_{\mu\nu}$ and the skew-symmetric vorticity tensor $\widetilde{F}_{\mu\nu}$. Tensors $g_{\mu\nu}$ and $\widetilde{F}_{\mu\nu}$ enter equations of motion as an effec-

tive metric tensor and an effective magnetic field respectively. Our results demonstrate a close relation of the many-body dynamics in Lagrangian frame to the quantum dynamics on curved manifolds.

We also derived local conservation laws for the number of particles and for momentum in the Lagrangian frame, and presented closed microscopic expressions for the stress tensor and for the corresponding stress force. The local momentum conservation law in the Lagrangian frame reduces to a zero force condition. The inertia forces exactly compensate the external force and the stress force in every point of the Lagrangian $\boldsymbol{\xi}$-space. The net force, exerted on every fluid element, is exactly zero, which results in zero current density and a time-independent density distribution. This property is the main advantage of the Lagrangian description. It suggests one of the most promising application of our formalism, which is a new reformulation of TDDFT in a form similar to the static theory. Indeed the main practical problem of TDDFT is an inevitable strong nonlocality of exchange correlation potentials[4,7,8]. The physical reason for this is just the nonadiabatic motion of fluid elements. When time is flowing, new and new fluid elements arrive at a given point $\mathbf{x}$, and bring an information about surrounding space, producing the above nonlocality. Using our reformulation of the many-body theory as a basis for TDDFT one can completely remove the very source on the nonlocality, which is of extreme practical importance. In this paper we did not touch these questions since it required an extended special consideration. A detailed formulation of TDDFT in the Lagrangian frame will be presented in the next paper of this series[30].

In this paper we also considered two illustrative examples of application. The most interesting of them is the description a nonlinear semiclassical dynamics of a collisionless Fermi gas. We have shown that the full problem can be separated into two independent parts. The first one is the solution of a universal kinetic problem, which defines the stress tensor as a universal functional of $g_{\mu\nu}$ and $F_{\mu\nu}$. This stress tensor is used as an input for the second "hydrodynamic" part of the problem, determining the dynamics of the velocity vector. This separation of the initial many-body problem can be viewed as a particular realization of TDDFT in the hydrodynamic formulation[4,22]. In the case of a fast long wavelength dynamics (similar to that for plasma oscillations), the universal kinetic problem can be solved explicitly. The solution is extremely simple – the Wigner function in the Lagrangian frame is time-independent. The corresponding "hydrodynamic" problem also can be formulated in the explicit form. It reduces to a closed nonlinear elasticity theory of a Fermi gas. This elasticity theory is, in fact, the Lagrangian formulation of the generalized hydrodynamics derived in Refs. 19,20. The generalized hydrodynamics proved to be useful in the description of a small-amplitude collective dynamics of an electron gas. It gives the correct (consistent with the kinetic treatment) dispersion of plasma waves in homogeneous systems[19,20]

and recovers the exact dispersion of the edge modes in a confined geometry[31]. The results for the standing plasma waves in a parabolically trapped electron gas are also quite reasonable[32,33]. The Lagrangian "elasticity theory" of a Fermi gas, derived in Sec. IVB, is structurally much more simple than the Eulerian formulation of Refs. 19,20. Therefore, we believe that it should provide a good basis for the description of nonlinear dynamical effects in inhomogeneous many-electron systems.

APPENDIX A: THE DIVERGENCE REPRESENTATION OF THE INTERACTION STRESS FORCE

By definition the infraction stress force $\widehat{\mathbf{F}}^{\mathrm{int}}$ is the commutator of the current operator $\widehat{\mathbf{j}}$ and the interaction Hamiltonian $\widehat{W}$,

$$\widehat{\mathbf{F}}^{\mathrm{int}}(\mathbf{x}) = im[\widehat{\mathbf{j}}(\mathbf{x}), \widehat{W}]_{-}. \qquad (A1)$$

In the coordinate representation operators $\widehat{\mathbf{j}}$ and $\widehat{W}$ are defined as follows

$$\widehat{\mathbf{j}}(\mathbf{x}) = \frac{1}{2m}\sum_{i}[\widehat{\mathbf{p}}_{i}, \delta(\mathbf{x}-\mathbf{x}_{i})]_{+}, \qquad (A2)$$

$$\widehat{W} = \frac{1}{2}\sum_{i,j} w(\mathbf{x}_{i},\mathbf{x}_{j}), \qquad (A3)$$

where i and j label particles. Calculating the commutator of Eq. (A1) we get the force in the following form

$$\widehat{\mathbf{F}}^{\mathrm{int}}(\mathbf{x}) = \frac{1}{2}\sum_{i,j}\left[\delta(\mathbf{x}-\mathbf{x}_{i})\frac{\partial w(\mathbf{x}_{i},\mathbf{x}_{j})}{\partial \mathbf{x}_{i}}\right.$$
$$\left. +\delta(\mathbf{x}-\mathbf{x}_{j})\frac{\partial w(\mathbf{x}_{i},\mathbf{x}_{j})}{\partial \mathbf{x}_{j}}\right]. \qquad (A4)$$

If the interaction potential satisfies the Newton's third law,

$$\frac{\partial w(\mathbf{x}_{i},\mathbf{x}_{j})}{\partial \mathbf{x}_{i}} = -\frac{\partial w(\mathbf{x}_{i},\mathbf{x}_{j})}{\partial \mathbf{x}_{j}}, \qquad (A5)$$

Eq. (A4) takes the form

$$\widehat{\mathbf{F}}^{\mathrm{int}}(\mathbf{x}) = \frac{1}{2}\sum_{i,j}\left[\delta(\mathbf{x}-\mathbf{x}_{i}) - \delta(\mathbf{x}-\mathbf{x}_{j})\right]\frac{\partial w(\mathbf{x}_{i},\mathbf{x}_{j})}{\partial \mathbf{x}_{i}}. \qquad (A6)$$

The difference of delta-functions in Eq. (A6) can be transformed as follows

$$\delta(\mathbf{x}-\mathbf{x}_{i}) - \delta(\mathbf{x}-\mathbf{x}_{j}) = \left[1 - e^{(\mathbf{x}_{i}-\mathbf{x}_{j})\frac{\partial}{\partial\mathbf{x}}}\right]\delta(\mathbf{x}-\mathbf{x}_{i})$$

$$= -(\mathbf{x}_{i}-\mathbf{x}_{j})\frac{\partial}{\partial\mathbf{x}}\int_{0}^{1} d\lambda e^{\lambda(\mathbf{x}_{i}-\mathbf{x}_{j})\frac{\partial}{\partial\mathbf{x}}}\delta(\mathbf{x}-\mathbf{x}_{i})$$

$$= -\frac{\partial}{\partial\mathbf{x}}(\mathbf{x}_{i}-\mathbf{x}_{j})\int_{0}^{1} d\lambda\,\delta(\mathbf{x}-\mathbf{x}_{i}-\lambda(\mathbf{x}_{j}-\mathbf{x}_{i})). \qquad (A7)$$

Inserting Eq. (A7) into Eq. (A6) we get the final representation for the interaction stress force

$$\widehat{F}^{\mathrm{int}}_{\mu}(\mathbf{x}) = \frac{\partial}{\partial x^{\nu}}\widehat{W}_{\mu\nu}(\mathbf{x}), \qquad (A8)$$

where $\widehat{W}_{\mu\nu}(\mathbf{x})$ is the interaction stress tensor operator

$$\widehat{W}_{\mu\nu}(\mathbf{x}) = -\frac{1}{2}\sum_{i,j}\int_{0}^{1} d\lambda\,\delta(\mathbf{x}-\mathbf{x}_{i}-\lambda(\mathbf{x}_{j}-\mathbf{x}_{i}))$$

$$\times(x^{\nu}_{i}-x^{\nu}_{j})\frac{\partial w(\mathbf{x}_{i},\mathbf{x}_{j})}{\partial x^{\mu}_{i}}. \qquad (A9)$$

The λ-integration in Eq. (A9) is along the line that connects two interacting particles.

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